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Reliability and maintainability theory

ON THE SEMISIMPLICITY OF TRANSITION RATE MATRICES

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1. INTRODUCTION

The semisimplicity of the transition matrix Φ in availability problems, and the fact that its eigenvalues are simple, is a result known in practice (1, p. 81), since these problems are modeled as monodesmic Markov processes. However, a detailed proof of this result is not available in current literature to the best of our knowledge.
As we show in this paper, in the stationary case, the semisimplicity of the transition rate matrix, Λ , implies the semisimplicity of the corresponding transition matrix, Φ ; we have not been able to prove a similar result for non stationary processes.
The purpose of this paper is to show that for both stationary and non stationary processes, the transition rate matrix pertinent to availability problems is semisimple.
We have been motivated to the research reported on here, by the need of a computer oriented formulation of the reliability problems, arisen in the C.A.R.A. (Computer Aided Reliability Analysis) Project at Selenia.
As far as Markov processes and matrix algebra are concerned, our terminology is consistent with Ref.s (1) and (2) respectively.

2. MATHEMATICAL BACKGROUND

A few definitions and theorems, available in the literature on matrix theory, are necessary as a mathematical background for our purpose. We collect the main of them in what follows.

Definition 2.1 (4, vol.II, p.50, def.2'): A matrix $\mathbf{A}$ is called reducible if there is a permutation that puts it into the form

$$\bar{A} = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

where $\mathbf{B}$ and $\mathbf{D}$ are square matrices. Otherwise $\mathbf{A}$ is called irreducible.

The irreducibility of a matrix has a geometrical interpretation in terms of graph theory. To show this we introduce the following

Definition 2.2 (5, p.19): Given an $n \times n$ matrix $\mathbf{A}$, a finite directed graph $G(\mathbf{A})$ can be associated with it, in the following way: consider in the plane n distinct points $P_1, \ldots, P_n$, which we call nodes. For every entry $a_{ij} \neq 0$ of $\mathbf{A}$ we connect P_i to P_j with a path $\overline{P_iP_j}$ directed from P_i to P_j. Each diagonal entry of $\mathbf{A}$ is associated with a path connecting the corresponding node to itself. These paths are called loops. The graph so obtained is a picture of the matrix structure and is referred to as the finite directed graph associated with the matrix.

Definition 2.3 (5, p.20, def.1.6): A directed graph is strongly connected if, for any ordered pair of nodes P_i and P_j, there exists a directed path

$$\overline{P_iP_k}, \overline{P_kP_l}, \overline{P_lP_m}, \ldots, \overline{P_qP_j}$$

connecting P_i to P_j.

The irreducibility of a matrix **A** is related to the strongly connection of its directed graph G(**A**), by the following

Theorem 2.1 (5, p.20, th.1.6): An n x n matrix **A** is irreducible if and only if its directed graph G(**A**) is strongly connected.

Corollary 2.1: If a square matrix **A** is irreducible, its transponse , $\mathbf{A}^T$, is irreducible.

We introduce now a number of classes of matrices.

Definition 2.4 (6, p.165, def.1.4) and (6, p.164, th. 1.3): A square matrix **A** belongs to the class P_0 if all its principal minors are nonnegative.

Definition 2.5 (7, p.386, def.4.1) : A square matrix **A** belongs to the class Z if its off-diagonal elements are all nonpositive. $\mathbf{A}^T$ belongs to the same class.

Definition 2.6 (7, p.391, def.5.2): A square matrix **A** belongs to the class K_0 if it belongs to the class Z and all its principal minors are nonnegative. That is

$$K_0 \triangleq Z \cap P_0 \tag{2.1}$$

Definition 2.7 (6, p.166, def. 2.3): A matrix **A** belongs to the class S_0 if there exists a vector $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$, such that $\mathbf{A}\,\mathbf{x} \geqslant \mathbf{0}$

Definition 2.8 (6, p.167, def.3.1): A matrix **A** (m x n) is said to be irreducibly S_0 if **A** belongs to S_0 and either n=1 or n>1 and no matrix obtained from **A** by omitting at least one column belongs to S_0 . The class of irreducibly S_0 matrices will be denoted by M .

Theorem 2.2 (6, p.165, 1.6): If $\mathbf{A} \in P_0$ then $\mathbf{A}^T \in P_0$.

Corollary 2.2 : If $\mathbf{A} \in K_0$ then $\mathbf{A}^T \in K_0$.

Proof: It follows directly from def. 2.6, def. 2.5 and th. 2.2.

Theorem 2.3 (6, p.166, th.2.6): If $\mathbf{A} \in P_0$ then $\mathbf{A} \in S_0$.

Theorem 2.4 (7, p.391, 5.6): Let $\mathbf{A} \in K_0$ be singular of order n and irreducible. Then **A** has rank (n-1) and there exists a vector $\mathbf{y} > 0$ such that $\mathbf{A}\,\mathbf{y} = 0$.

Theorem 2.5 (6, p.170, th.3.10) : Let **A** be an m x n matrix of rank h. Suppose that $\mathbf{A} \in S_0$, then

1. the matrix $\mathbf{A} \in M$ and h=n-1
2. both **A** and - **A** belong to M
3. there exist positive vectors **u** and **x** (of lenght m and n respectively) such that $\mathbf{u}^T\mathbf{A} = 0$ and $\mathbf{A}\,\mathbf{x} = 0$ and h=n-1.

Theorem 2.6 (6, p.171, th.3.13): Let **A** be a square matrix of order n>1, then the following three conditions are equivalent:

1. both **A** and -**A** belong to M
2. $\mathbf{A} \in M$ and **A** is singular (det **A** =0)
3. **A** is singular and adj **A** is either positive or negative.

Theorem 2.7 (8, p.266, th.2.5): If α is an eigenvalue of **A** then $(\alpha + x)$ is an eigenvalue of $\mathbf{B} = \mathbf{A} + x\,\mathbf{I}$.

Definition 2.9 (2, p.76) : If an n x n matrix **A** has a total of n linearly indipendent eigenvectors, regardless of degeneracy, then **A** is said to be semisimple.

Theorem 2.8 (4, vol.I, p.72): Eigenvectors belonging to pairwise distinct eigenvalues are always linearly indipendent.

Definition 2.10 (9, p.63): The set of all eigenvalues of a square matrix is known as its spectrum.

Theorem 2.9 (9, p.173, th. 5.3.4): If $\mu_1, \ldots, \mu_n$ are the eigenvalues of any complex $n \times n$ matrix $\mathbf{A}$ and f is a complex valued function defined on the spectrum of $\mathbf{A}$ then the eigenvalues of $f(\mathbf{A})$ are $f(\mu_1), \ldots, f(\mu_n)$.

3. MAIN RESULTS

According to def. 2.9 and th. 2.8, in order to prove the semisimplicity of the transition rate matrix Λ, it is enough for us to prove that its eigenvalues are simple. To this purpose we use the following

Theorem 3.1: Let $\mathbf{M}$ be any real $n \times n$ matrix; let α be one of its eigenvalues and $\mathbf{v}$ the eigenvector associated with it. A sufficient condition for α to be simple is that one of the following sets of conditions is satisfied:

$$\begin{cases} \mathbf{v} > 0 \\ \operatorname{adj}(\alpha \mathbf{I} - \mathbf{M}) > 0 \end{cases} ; \quad \begin{cases} \mathbf{v} > 0 \\ \operatorname{adj}(\alpha \mathbf{I} - \mathbf{M}) < 0 \end{cases} , \quad \begin{cases} \mathbf{v} < 0 \\ \operatorname{adj}(\alpha \mathbf{I} - \mathbf{M}) > 0 \end{cases} ; \quad \begin{cases} \mathbf{v} < 0 \\ \operatorname{adj}(\alpha \mathbf{I} - \mathbf{M}) < 0 \end{cases} \tag{3.1}$$

Proof: The following identity is true for all x numbers real or complex, including eigenvalues of $\mathbf{M}$:

$$[\operatorname{adj}(x\mathbf{I} - \mathbf{M})](x\mathbf{I} - \mathbf{M}) = \det(x\mathbf{I} - \mathbf{M})\,\mathbf{I} \tag{3.2}$$

For x not an eigenvalue of $\mathbf{M}$, eq. (3.2) is clear since, in this case, $\det(x\mathbf{I} - \mathbf{M}) \neq 0$ and eq. (3.2) follows directly by the operative definition of inverse matrix; when x is an eigenvalue of $\mathbf{M}$, eq. (3.2) follows by continuity; in any case it is proven in (8, p.402, th.2.10).

Let

$$d(x) \triangleq \det(x\mathbf{I} - \mathbf{M}) \tag{3.3}$$

Then, differentiating eq. (3.2) with respect to x, we get

$$\frac{d}{dx}[\operatorname{adj}(x\mathbf{I} - \mathbf{M})](x\mathbf{I} - \mathbf{M}) + \operatorname{adj}(x\mathbf{I} - \mathbf{M}) = d'(x)\,\mathbf{I} \tag{3.4}$$

By definition of eigenvector $\mathbf{v}$ corresponding to a given eigenvalue α of a matrix the following equation holds

$$\mathbf{M}\mathbf{v} = \alpha\,\mathbf{v} \tag{3.5}$$

from which

$$(\alpha\mathbf{I} - \mathbf{M})\,\mathbf{v} = 0 \tag{3.6}$$

If in eq. (3.4) we put $x = \alpha$ and postmultiply by $\mathbf{v}$, we get

$$\frac{d}{dx}[\operatorname{adj}(\alpha\mathbf{I} - \mathbf{M})](\alpha\mathbf{I} - \mathbf{M})\mathbf{v} + \operatorname{adj}(\alpha\mathbf{I} - \mathbf{M})\mathbf{v} = d'(\alpha)\,\mathbf{I}\,\mathbf{v} \tag{3.7}$$

and, by (3.6), eq. (3.7) becomes

$$\operatorname{adj}(\alpha\mathbf{I} - \mathbf{M})\,\mathbf{v} = d'(\alpha)\,\mathbf{I}\,\mathbf{v} \tag{3.8}$$

If one of the sets of conditions of the theorem is satisfied, eq. (3.8) implies

$$d'(\alpha)\,\mathbf{I}\,\mathbf{v} \neq 0 \tag{3.9}$$

and, being $\mathbf{v} > 0$ or $\mathbf{v} < 0$, eq. (3.9) implies

$$d'(\alpha) \neq 0 \tag{3.10}$$

Eq. (3.10) is true if and only if α is a simple eigenvalue. ▮

Theorem 3.2 : The opposite of a transition rate matrix, $-\Lambda$, belongs to the class K_o

$$-\Lambda \in K_o \tag{3.11}$$

Proof: The proof is given in a separate paper (3). $\blacksquare$

Theorem 3.3 : The opposite of a transition rate matrix, $-\Lambda$, belongs to the class S_o

$$-\Lambda \in S_o \tag{3.12}$$

Proof: Since $-\Lambda \in K_o$, def. 2.6 implies $-\Lambda \in P_o$ and by th.2.3 our proposition follows. $\blacksquare$

Theorem 3.4 : The opposite of an $n \times n$ irreducible transition rate matrix has rank $(n-1)$.

Proof: It is well known that any transition rate matrix has column sums equal to zero, then it is singular. Due to this, to th.3.2 and to the assumed irreducibility, th.2.4 applies. $\blacksquare$

Theorem 3.5 : Both an irreducible transition rate matrix, Λ, and its opposite, $-\Lambda$, belong to class M.

Proof: In force of th.s 3.3 and 3.4, $-\Lambda \in S_o$ and it has rank $(n-1)$. Due to the point 3 of th.2.5, what it remains to be proven is that there exist positive solutions for the following two problems

$$-\Lambda \mathbf{x} = 0 \tag{3.13}$$

$$-\Lambda^T \mathbf{u} = 0 \tag{3.14}$$

Being $-\Lambda \in K_o$ and irreducible, the same holds for $-\Lambda^T$ in force of coro.s 2.1 and 2.2. Then th.2.4 applies. Condition 3 of th.2.5 is satisfied and than the condition 2, that proves our theorem. $\blacksquare$

Theorem 3.6 : The adjoint of the opposite of an irreducible transition rate matrix is either positive or negative

$$\text{adj}(-\Lambda) > 0 \qquad \text{or} \qquad \text{adj}(-\Lambda) < 0 \tag{3.15}$$

Proof: In any reliability problem, Λ matrix has order $\geqslant 2$. Due to th.3.5, th.2.6 applies. $\blacksquare$

Let's now prove the following

Theorem 3.7 (Fundamental): An irreducible transition rate matrix has simple eigenvalues.

Proof: Let's consider the matrix

$$\mathbf{M} \triangleq \Lambda + x\,\mathbf{I} \tag{3.16}$$

where x is any number. If α is any eigenvalue of Λ, th.2.7 states that $(\alpha + x)$ is the corresponding eigenvalue of $\mathbf{M}$; then to prove that the eigenvalues of Λ are simple, it is sufficient to prove that the eigenvalues of $\mathbf{M}$ are simple.

From eq. (3.16) we have

$$-\Lambda = x\,\mathbf{I} - \mathbf{M} \tag{3.17}$$

then, by th.3.6

$$\text{either} \quad \text{adj}(x\,\mathbf{I} - \mathbf{M}) > 0 \qquad \text{or} \qquad \text{adj}(x\,\mathbf{I} - \mathbf{M}) < 0 \tag{3.18}$$

If β is an eigenvalue of $\mathbf{M}$, with $\mathbf{u}$ as corresponding eigenvector, by definition we have

$$\mathbf{M}\mathbf{u} = \beta\,\mathbf{u} \tag{3.19}$$

from which

$$(\mathbf{M} - \beta\,\mathbf{I})\,\mathbf{u} = 0 \tag{3.20}$$

Substituting eq. (3.16) into (3.20), we get

$$(\Lambda + x \, \mathbf{I} - \beta \, \mathbf{I}) \quad = 0, \quad \forall x \tag{3.21}$$

and taking $x = \beta$, eq.(3.21) becomes

$$\Lambda \mathbf{u} = 0 \qquad \text{or} \qquad - \Lambda \mathbf{u} = 0 \tag{3.22}$$

Being $-\Lambda \in K_0$ and irreducible, th.2.4 states that there exists a solution $\mathbf{u} > 0$ to eq.(3.22). The above result and equation (3.18) are the sufficient conditions requi red by th.3.1 for an eigenvalue β to be simple. We can state the same for any β. Then we conclude that each eigenvalue of **M** is simple. ▮

We try now to extend the results obtained for the Λ matrix to the transition matrix Φ (t).

Theorem 3.8: For a stationary Markov process, the semisimplicity of the transition rate matrix, Λ, implies the semisimplicity of the pertinent transition matrix, Φ (t).
Proof: For a stationary Markov process (1, p.776)

$$\Phi(t) = \exp(\Lambda t) \tag{3.23}$$

Our theorem is a simple consequence of th.2.9. ▮

If we restrict our considerations to transition rate matrices, we can notice that all arguments do not consider explicitly any time dependence of Λ itself. Then we can repeat them for each instant of time, if Λ is time dependent, and get the same results. In particular, the fundamental th. 3.7 holds also for non stationary problems: in other words Λ's eigenvalues can vary with time, but in any case they are simple.

If we try to extend this result to the transition matrix, Φ (t), we find that, for a non stationary process, a relationship like eq.(3.23) is not available in general; much mo re, a closed relationship does not exist in the general case. So we have not been able to prove any theorem similar to th.3.8 for a non stationary Markov process.

4. CONCLUDING REMARKS

All the results of the previous section share an algebraic flavour. This is in accordance with the approach we have favored.
Among the various formal properties of a transition rate matrix that we have used, there is one, irreducibility, that needs an interpretation in terms of reliability theory.
We start by giving the following

Definition 4.1: The availability (A) of a system is defined as the probability that at a time t the system is correctly working, indipendently from the fact that before that time it has failed and has been subjected to repairs.

Now we can prove the following

Theorem 4.1: The transition rate matrix of any availability problem is irreducible.
Proof: According to def.4.1, it appears that, for the case of availability, it is possible to reach the all-unfailed state starting from any other state of the system, either of correct working or of failure. From the all-unfailed state it is possible to go by failures to any other state; then, according to def.2.3, the graph associated with the transition rate matrix is strongly connected and, by th. 2.1, the matrix is irreducible. ▮

Our general conclusion is than:

"The transition rate matrix of any availability problem is semisimple.

The transition matrix of any stationary availability problem is semisimple".

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ON THE SEMISIMPLICITY OF TRANSITION RATE MATRICES

Abstract: A common assumption in reliability theory for electronic systems is the validity of the Poisson model for components failures and repairs. Under this assumption it is well known that any electronic system can be modeled as a homogeneous Markov process (1), i.e. its transition rate matrix is not time dependent. It is the purpose of this paper to consider the problem of the order of multiplicity of transition rate matrix eigenvalues. We show that they are simple, and then the transition rate matrix is semisimple (2), if the transition rate matrix is irreducible. In the reliability language that means that any availability problem has simple eigenvalues and semisimple transition rate matrix. This property is particularly relevant and has important implications on theoretical studies and computational techniques. Our proof is fully algebraic and relies heavely on our previous results, namely on the fact that the opposite of a transition rate matrix for reliability problems is a K_0 matrix (3). Altough we consider in detail the case of a homogeneous Markov process in this paper, our arguments can be easely extended to the case of inhomogeneous Markov processes, and of time dependent transition rate matrices.

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