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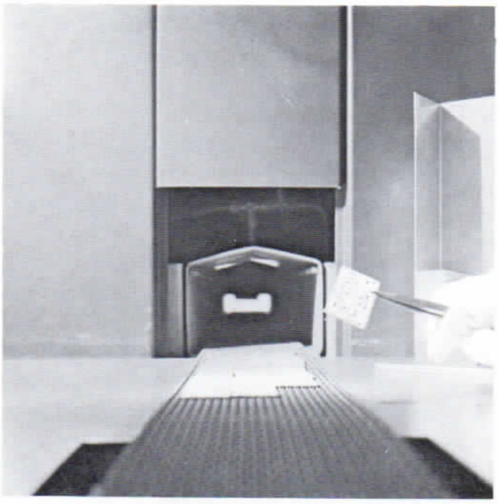
SUMMARY



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Edited by Selenia,
Industrie Elettroniche Associate S.p.A.
Via Tiburtina km 12.400 – 00131 Rome, Italy

CALCULATION OF RELIABILITY WHEN FAILURE RATE FIGURE INCREASES WITH TIME

ERNANI CARRADA – ROBERTO SOMMA
Reliability Dept.

SUMMARY

Given the shortcomings of the Poisson model in describing the reliability of systems and components subject to wear, the Weibull model with m = 1 has been considered. For this model, general formulae have been developed for the calculation of the reliability of single components and of series configurations. For configurations in operative or standby redundancy, less general formulae have been developed.

These formulae are easily programmable, even with small computers.

The formulae obtained also permit approximate calculation of reliability of components and series configurations with any kind of ageing law. This model can, in fact, always be approximated by suitable sequences of straight line segments.

1. Introduction

In the electronics field, the Poisson model is widely used and appreciated. This model consists of the assumptions that the failure rate of a certain population of components is constant in time, that the failures are statistically independent of each other, and that the probability of more than one failure in a very small time interval dt is zero. In spite of the criticism made of this model, it is universally adopted, and is in fact the standard basis for the reliability calculations of electronic systems; this is justified by the ease of calculation, which at times is an indispensable factor for the calculations performance itself.

Among the cases in which the above simple approximation is insufficient to find the solution to specific problems, the example of systems with wear phenomena which cause an increase of the failure rate in time is particularly interesting.

An example of such a problem is the launch phase reliability calculation of the gyroscope of a missile guidance system, which has been subject to wear during the acceptance tests and the periodic preventive maintenance tests performed on it at regular time intervals during its storage in the arsenal.

This article is devoted to populations of components for which the failure rate increases linearly in time, and formulae are developed for the reliability calculation of systems with simple configurations.

2. Poisson model

For cases in which the failure rate is constant in time, the general formula

$$R(t_m = t_f - t_i) = exp\ [-\int_{t_i}^{t_f} z(t)\,dt] \qquad (1)$$

where: t_i = mission start time
t_f = mission end time

becomes:

$$R(t_m) = exp\,(-\lambda t_m) \qquad (2)$$

From these formulae it appears that if a component is correctly working at the start of the mission, the success probability during the mission time is independent of the age of the component at mission start. This is shown by Fig. 1.

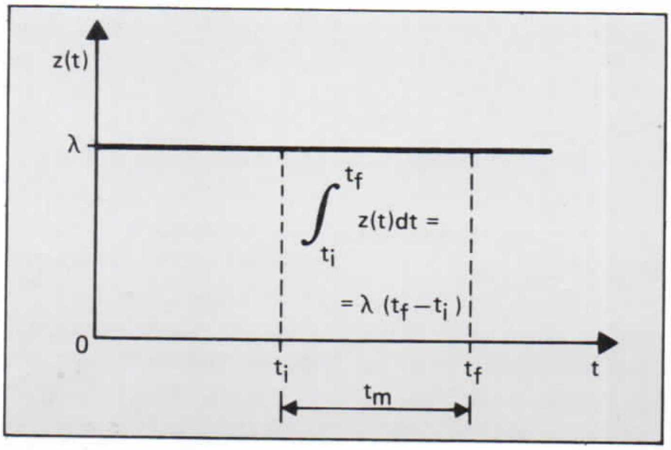


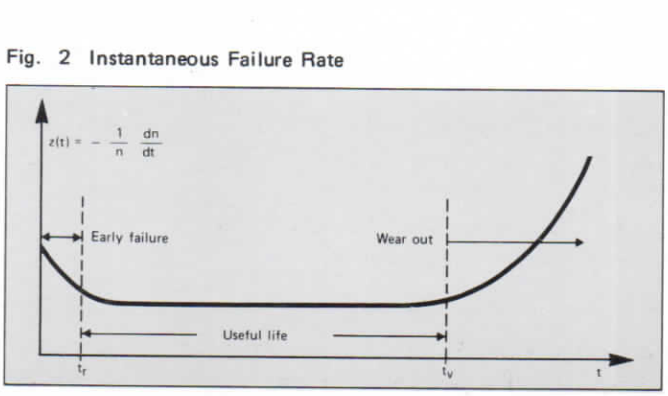
Fig. 1 Independence of $R(t_m)$ from t_i

For this reason, the general formula (1) for the reliability of a mission of duration equal to t_m is normally expressed as follows:

$$R(t_m) = exp\ [-\int_0^{t_m} z(t)dt]$$

The use of the Poisson model in the reliability calculations of electronics systems is, as is well known, based on the fact that, with a good approximation, the failure rate of the electronic and electromechanical components has a time dependency of the type shown in Fig. 2.

Fig. 2 Instantaneous Failure Rate



An artificial population with a failure rate independent of time for a given mission can be obtained by elimination, through screening, of the intrinsically weak components which contribute to early failure, and by consideration of mission times which are shorter than the useful life of the components; or by substitution of the worn out components by screened ones.

The possibility of eliminating by means of screening the inevitable early failures permits the construction of a general model of the instantaneous failure rate as shown in Fig. 3.

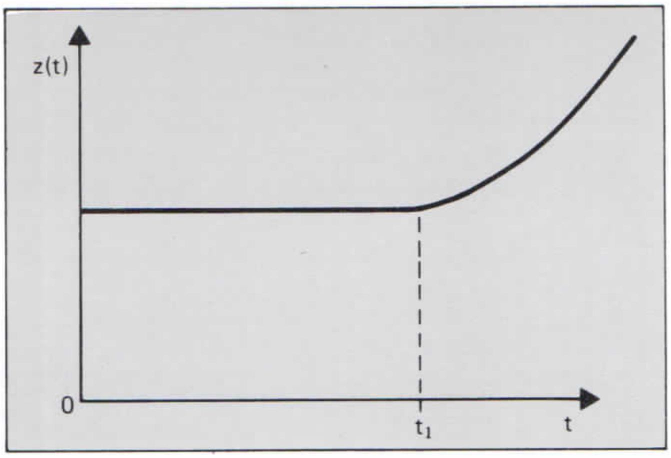


Fig. 3 Schematic of Instantaneous Failure Rate after Screening

Time start coincides with the end of the screening. From the Figure it can be first seen that there is a time interval whose duration (not yet experimentally evaluated for some semiconductor components, and practically negligible for some electromechanical components) depends on the type of component; during this time interval, the failure rate is constant. Following this interval, the consequences of wear appear, and the failure rate increases with time. The following paragraph explains this last case in detail.

3. Weibull model with m = 1

3.1. COMPONENT RELIABILITY

The simplest hypothesis which can be formulated to take into account the consequences of wear is to assume an instantaneous failure rate linearly increasing in time.

(3) $$z(t) = kt$$

As is well known, (3) is a particular case (m=1) of the more general

$$z(t) = kt^m$$

which represents the behavior in time of the instantaneous failure rate of a population of components whose life duration is given by the probability density function:

$$f(t) = kt^m \, exp\,[-\frac{k}{m+1}\, t^{(m+1)}]$$

known as the "Weibull distribution".

When the mission start coincides with the component life start, i.e. $t_i = 0$ and $t_f = t_m$, formula (1) becomes

(4) $$R(t_m) = exp\,(-\frac{k t_m^2}{2})$$

which is well known from literature [1].

The implicit assumption in (3) that $z(t) = 0$ for $t = 0$ appears unsuitable for practical purposes, and is, in any case, insufficient for the model given in Fig. 3. It is therefore assumed that

$$z(t) = \begin{cases} = \lambda_o & \text{for } t < t_1 \\ = \lambda_o + k(t - t_1) & \text{for } t > t_1 \end{cases}$$

i.e.

$$z(t) = \lambda_o + k(t - t_1) \cdot u(t - t_1)$$

where $u(t-t_1)$ is the known step function equal to zero for $t < t_1$ and equal to one for $t > t_1$.

Formula (4) thus becomes

(5) $$R(t_m) = exp\left\{-[\lambda_o t_m + \frac{1}{2}k(t_m - t_1)^2 \cdot u(t_m - t_1)]\right\} \quad \text{for } t_m \neq t_1$$

where: t_1 = instant in which the failure rate starts to increase.

The case in which $t_m = t_1$ is banal.

Formula (5) can be obtained either by analysis or by use of formula (1), taking into account Fig. 4.

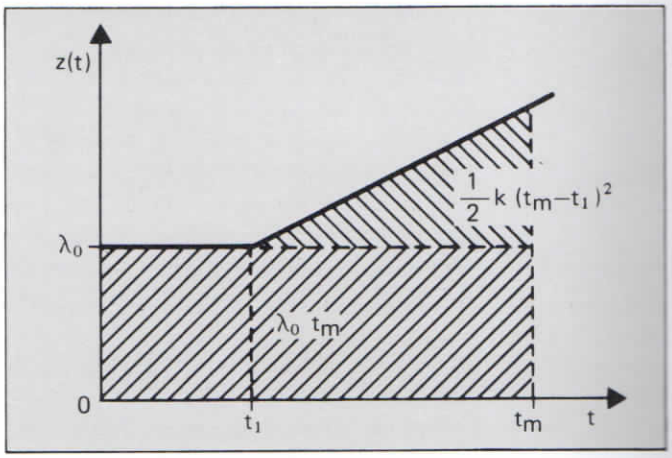


Fig. 4 Justification of Formula (5) exponent

Formula (5) is only valid where the mission time t_m is the true component life duration; if the mission starts when the component has already aged, one has:

(6) $R(t_m = t_f - t_i) =$

$$= exp\left\{-\left\{\lambda_o + [k(t_i - t_1) + \frac{1}{2} k t_m]\, u(t_i - t_1)\right\} t_m - \frac{1}{2} k (t_f - t_1)^2 \cdot u(t_f - t_1) \cdot u(t_1 - t_i)\right\}$$

with the condition

$$u(t_i - t_1) = \begin{cases} 0 & \text{for } t_i \leqslant t_1 \\ 1 & \text{for } t_i > t_1 \end{cases}$$

$$u(t_1 - t_i) = \begin{cases} 0 & \text{for } t_1 < t_i \\ 1 & \text{for } t_1 \geqslant t_i \end{cases}$$

The detailed calculation is given in Appendix A.

In the above formula, the term $k(t_i - t_1)$ represents the memorization of wear previous to mission start.

If the k coefficient appearing in the instantaneous failure rate expression is not constant in time, because of varying stresses for example, $z(t)$ takes on the form

(7) $$z(t) = \lambda_o + \sum_{s=1}^{n} (k_s - k_{(s-1)})(t - t_s) \cdot u(t - t_s)$$

where n is the number of points in which there is a slope change following the start, and t_s are the instants in which a slope change occurs. When the mission starts at $t = 0$, we have:

(8) $R(t_m) =$

$$= exp\left\{-[\lambda_o t_m + \frac{1}{2} \sum_{s=1}^{n} (k_s - k_{(s-1)})(t_m - t_s)^2 \cdot u(t_m - t_s)]\right\}$$

Formula (8) can be generalized for the case in which the mission starts at the generic instant t_i of the component life, and we obtain:

(9) $R(t_m = t_f - t_i) =$

$$= exp\left\{-\left\{\lambda_o t_m + \frac{1}{2} \sum_{s=1}^{n} (k_s - k_{(s-1)})(t_f - t_s)^2 \cdot u(t_s - t_i) \cdot u(t_f - t_s) + \sum_{s=1}^{n} [(k_s - k_{(s-1)})(t_i - t_s) + \frac{1}{2} (k_s - k_{(s-1)}) t_m]\, t_m \cdot u(t_i - t_s)\right\}\right\},$$

with the condition

$$u(t_i - t_s) = \begin{cases} 0 & \text{for } t_i \leqslant t_s \\ 1 & \text{for } t_i > t_s \end{cases}$$

$$u(t_s - t_i) = \begin{cases} 0 & \text{for } t_s < t_i \\ 1 & \text{for } t_s \geqslant t_i \end{cases}$$

The detailed calculations are contained in Appendix B.

The form chosen for the formulae (7), (8) and (9) makes them easily programmable also on small computers (as, for example, the HP 9820). Due to the fact that these calculations are very rapid, formula (9) can be used for approximate reliability calculations independently of the behavior of $z(t)$. In fact, the $z(t)$ behavior can always be approximated by a sequence of straight line segments.

3.2. SERIES CONFIGURATION

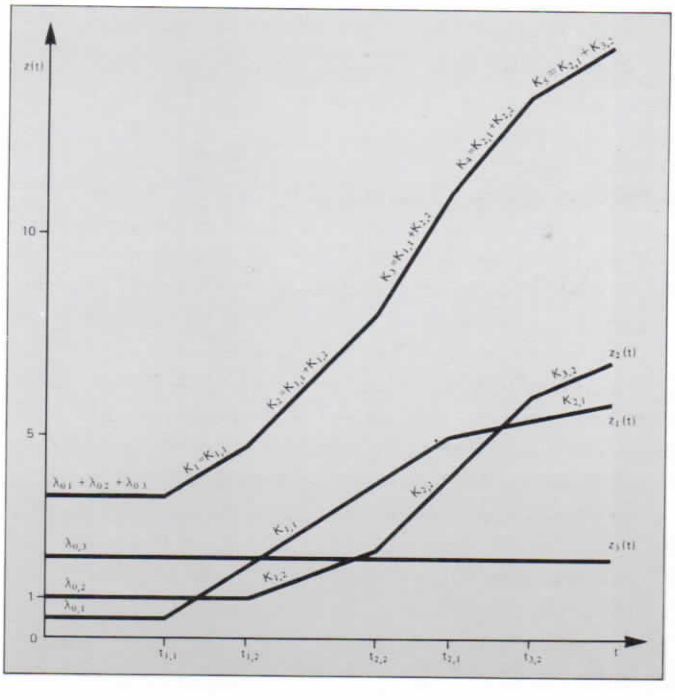
If a system is composed of n components, the success of each of which is a condition for overall system success (i.e. components are series connected in the reliability model), the system reliability for a mission of duration t_m is:

(10) $$R_s(t_m = t_f - t_i) = \prod_{j=1}^{n} R_j(t_m)$$

when the h events "i^{th} component success" are statistically independent of each other.

Since the product of the exponentials is equal to the exponential having as exponent the sum of the exponents, and since the sum of integrals having the same integration extremes is equal to an integral between the same extremes of the sum of the integrands, in order to solve (10) it is sufficient to perform the sum of $z_j(t)$ and insert this sum in the most suitable of the previous formulae.

Fig. 5 Construction of the sum of $z_j(t)$



Consideration of Fig. 5 suggests a simple graphic method of performing the sum of $z_j(t)$; it is, in fact, sufficient, as shown in Fig. 6, to mark on a time axis the instants at which the various $z_j(t)$ change slope, to assign a new orderly number to the instants on the new axis, and, for each of the new intervals thus created, to perform the sum of the $k_{i,j}$ relating to the single intervals; for the start time, the sum of $\lambda_{0,j}$ must be performed. It follows that

$$\sum_{j=1}^{n} z_j(t) = \sum_{j=1}^{n} \lambda_{0,j} + \sum_{s'=1}^{n'} (k'_{s'} - k'_{(s'-1)}) (t - t_{s'}) \, u(t - t_{s'})$$

where n' is the number of slope change points on the new time axis, and the generic $k'_{s'}$ is the sum of the $k_{i,j}$ coefficients which are in the interval following $t_{s'}$.

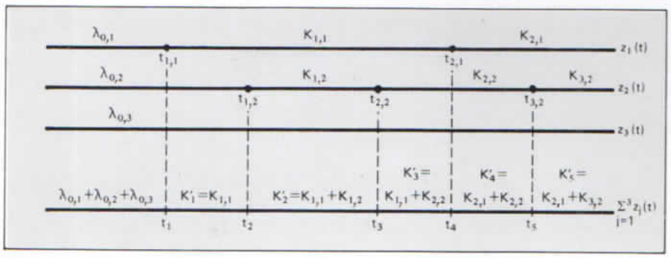


Fig. 6 Graphic Method of Evaluation of the Sum of $z_j(t)$

The reliability calculation of a series system is thus reduced to that of a single component, already examined above.

3.3. OPERATIVE REDUNDANCY

When the reliability of a component is insufficient for a given mission, more components, each of them able to meet the necessary performance requirements, can be interconnected in order to obtain the required reliability, i.e. a redundancy can be performed.

If all the system components function simultaneously, and system failure is only caused by the failure of all of them, the redundancy is said to be "operative", and the system is represented by the reliability block diagram of Fig. 7.

As an example, a system for the lifting of liquid can be considered, which consists of 2 identical pumps, each capable of performing the required function, but which normally divide the work equally between them.

If the instantaneous failure rate of the considered components linearly increases in time, it is reasonable to assume that the pump which performs all the work, following breakdown of the other pump, has an instantaneous failure rate which increases at a higher rate than when it performed only half of the work (see Fig. 8).

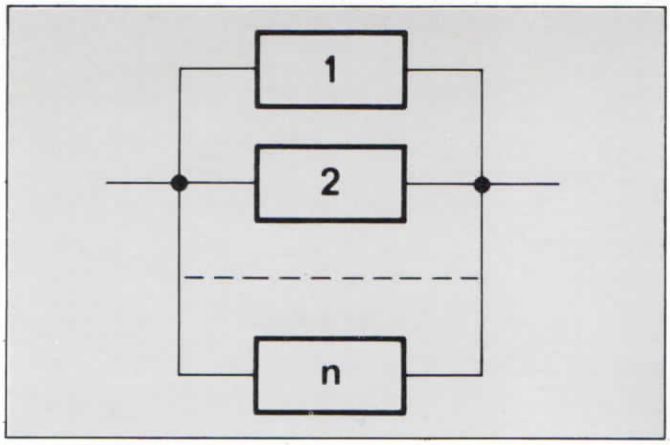


Fig. 7 Operative Redundancy Reliability Block Diagram

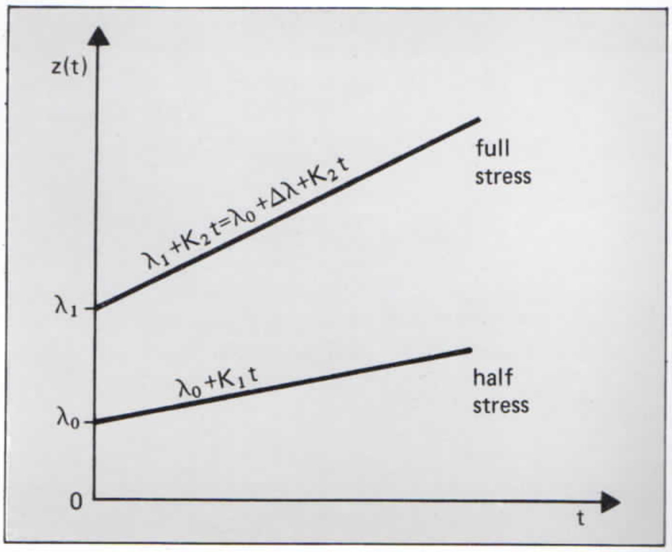
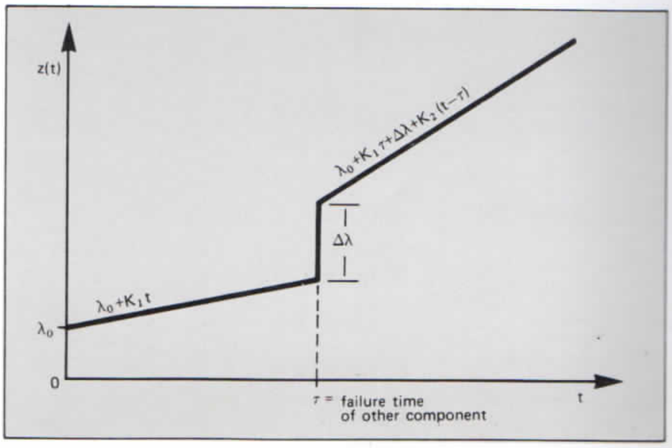


Fig. 8 Behavior of z(t) for Two Different Stress Conditions

The result of this hypothesis is that the instantaneous failure rate of each redunded component has the behavior shown in Fig. 9.

If the two pumps are indicated by A and B, the system success for the lifting at time t is ensured if one of the

Fig. 9 Failure Rate of a Component in Operative Redundancy



following mutually exclusive events takes place:

Event	*Probability*
a) Both components correctly work at time t	$[R_1(t)]^2$
b) Component A works until τ,	$R_1(\tau)$
fails at $(\tau, \tau + d\tau)$;	$dQ_1(\tau)$
Component B works until τ,	$R_1(\tau)$
continues to work until t;	$R_2(t-\tau)$
c) As b), changing A and B	as for b)

The following probabilities relate to the above events:

$$Pr(a) = [R_1(t)]^2 = exp[-(2\lambda_o t + k_1 t^2)]$$

$$Pr(b) = Pr(c) = R_1(\tau) \cdot dQ_1(\tau) \cdot R_1(\tau) \cdot R_2(t-\tau)$$

since the events b) and c) are statistically independent.

The following expressions relate to the single factors of the above formula:

$$R_1(\tau) = exp[-(\lambda_o \tau + \frac{k_1}{2}\tau^2)]$$

$$R_1(\tau)dQ_1(\tau) = f_1(\tau)d\tau = -\frac{dR_1(\tau)}{d\tau} \cdot d\tau = (\lambda_o + k_1\tau)\, exp[-(\lambda_o\tau + \frac{k_1}{2}\tau^2)]$$

$$R_2(t-\tau) = exp[-\int_0^{t-\tau}[\lambda_o + \Delta\lambda) + k_1\tau + k_2(t-\tau)]\, d(t-\tau)] = exp[-[(\lambda_o + \Delta\lambda + k_1\tau)(t-\tau) + \frac{k_2}{2}(t-\tau)^2]]$$

Since:

$$R_{op}(t) = Pr(a) + Pr(b) + Pr(c)$$

it follows that

$$R_{op}(t) = [R_1(t)]^2 + 2\int_0^t f_1(\tau) \cdot R_1(\tau) \cdot R_2(t-\tau)\, d\tau$$

The substitution in this latter formula of the above expressions and the calculation of the integral are somewhat difficult, and are given in Appendix C. The final expression, which permits the calculation of the success probability of the operative redundant configuration, is:

$$(11)\quad R_{op}(t) = (1 - \frac{2k_1}{k_2})\, exp[-(2\lambda_o t + k_1 t^2)] + \frac{2k_1}{k_2} exp\left\{-[(\lambda_o + \Delta\lambda)t + \frac{k_2}{2}t^2]\right\} + [k_1\Delta\lambda + (k_2 - k_1)(\lambda_o + k_1 t)] \cdot \sqrt{\frac{2\pi}{k_2^3}} \cdot exp[-\frac{(2k_2 - k_1)(2\lambda_o t + k_1 t^2)}{2k_2} - \frac{k_1 \Delta\lambda t}{k_2} + \frac{(\lambda_o - \Delta\lambda)^2}{2k_2}] \cdot [\Phi(\frac{\lambda_o - \Delta\lambda + k_1 t}{\sqrt{2k_2}}) - \Phi(\frac{\lambda_o - \Delta\lambda + (k_1 - k_2)t}{\sqrt{2k_2}})]$$

where:

$$\Phi(x) = ERF(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(t^2)\, dt,$$

is the known "error function" which is tabulated in mathematical and statistical manuals.

For calculation of this function, one of the following series can be used [2], [3]:

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)}}{k!(2k+1)} = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)} x^{(2k-1)}}{(2k-1)\cdot(k-1)!} = \frac{2}{\sqrt{\pi}} exp(-x^2) \sum_{k=0}^{\infty} \frac{2^k x^{(2k+1)}}{(2k+1)!!}$$

where

$$(2k+1)!! = 1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot (2k+1).$$

It is easily verified by means of (11) that $R_{op}(t=0) = 1$, and that when

$$\Delta\lambda = 0, \qquad k_1 = k_2,$$

formula (11) gives the known formula

$$R_{op}(t) = 2R(t) - [R(t)]^2$$

As an example of the size of the parameters in formula (11), the following can be noted.

A ball bearing is a very simple component, the failure rate of which can be assumed to depend only on wear. Its behavior is therefore representable by formula (3). By substitution in the known general formula

$$MTTF = \int_0^{\infty} R(t)\, dt,$$

of expression (4), the following relationship between MTTF (mean time to failure) and the k coefficient is obtained:

$$k = \frac{\pi}{2\,(MTTF)^2}$$

$k_1 = 1.6 \cdot 10^{-8}$ is related to a mean life of 10,000 hours, with a given stress; if the stress is doubled, and if the hypothesis [4] is accepted that the mean life is inversely proportional to the cube of the stress, $k_2 = 8k_1 \cong 1.3 \cdot 10^{-7}$ is obtained.

An electric machine is a complex component whose failure rate depends not only on parts subject to wear (e.g.

bearings, brushes), but also on parts (e.g. windings) for which $z(t) = \text{constant}$ can be assumed; its instantaneous failure rate behavior is therefore that shown in Fig. 4, with λ_o of the order of 10^{-7} [5]. It can be noted that if k of the order of 10^{-8} and λ_o of the order of 10^{-7} are assumed, after only 10 hours the contribution to the total failure rate due to wear is of the same order as λ_o.

It follows that, for short mission times or for unused wearable parts, λ_o must be taken into account, while this constant value can be ignored for long mission times (one year consists of 8760 hours) or for missions of short duration which take place after a certain number of tests, maintenance actions or working cycles, so that the contribution made by wearable parts is greater than the λ_o figure.

Fig. 10 shows the behavior of $R_{op}(t)$ in the simple case in which

$k_1 = 10^{-8}$; $\lambda_o = 10^{-7}$; $\Delta\lambda = 4 \cdot 10^{-7}$;

$k_2 = 10^{-8}$, i.e., wear is independent of stress;

$k_2 = 4 \cdot 10^{-8}$, i.e., the mean life is inversely proportional to the square of the stress;

$k_2 = 8 \cdot 10^{-8}$, i.e., the mean life is inversely proportional to the cube of the stress.

In the same Figure, the behavior of

$$R(t) = e^{-\lambda t - \frac{k}{2} t^2}$$

is shown as reference, related to only one non-redundant block, for which the following figures must obviously be assumed:

$$\lambda = \lambda_o + \Delta\lambda = 5 \cdot 10^{-7}$$

$$k = k_2 = 10^{-8},\ 4 \cdot 10^{-8},\ 8 \cdot 10^{-8}$$

which are relevant to full stress conditions.

3.4. STANDBY REDUNDANCY

A configuration in which the redundant components foreseen for a given mission are interconnected, so that only one of them is powered and working while the others are inactive, is called a "standby redundancy", and the system is represented by the model shown in Fig. 11.

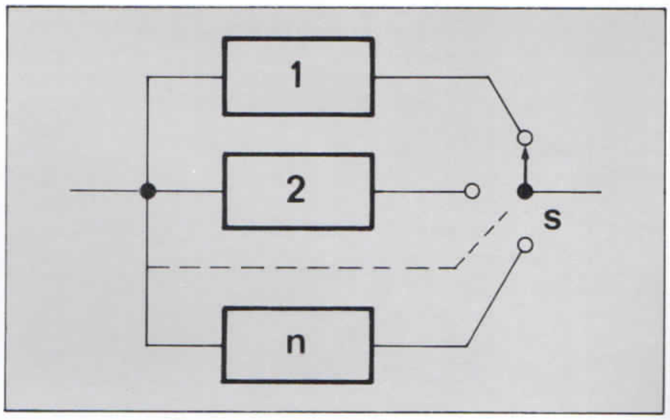


Fig. 11 Standby Redundancy Reliability Block Diagram

The S switch is an ideal component, capable of immediately realizing the failure of the working component and

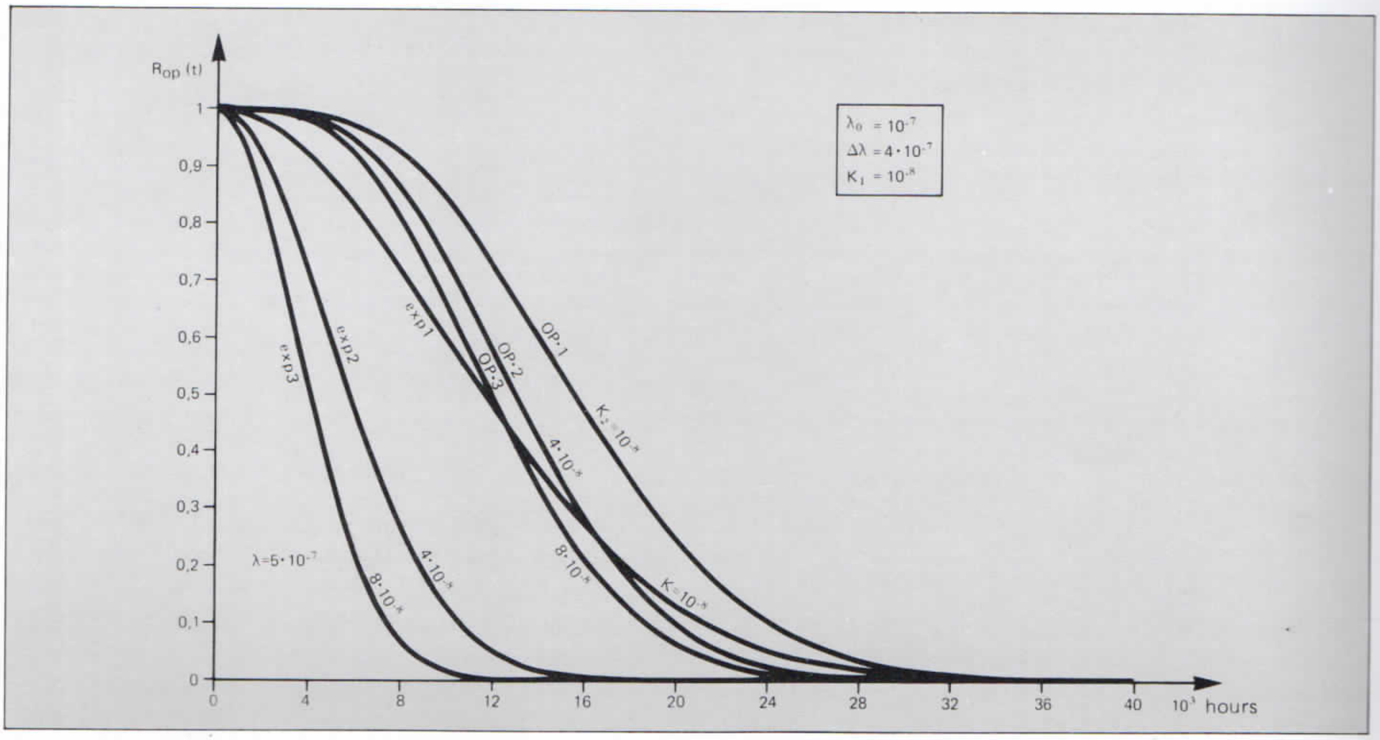


Fig. 10 Operative Redundancy Configuration Reliability Behavior

of inserting in line one of the other components able to correctly function.

In the case of only 2 components, one is subjected to all the stress while the other is completely unstressed. It seems reasonable to attribute to the latter component an instantaneous failure rate which increases very slowly in time, and the behavior of z(t) can be drawn as shown in Fig. 12.

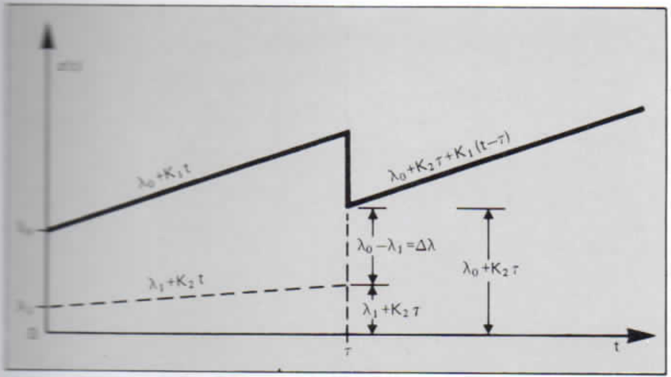


Fig. 12 Schematic of z(t) for Standby Redundancy

The system success at time t is obtained if at least one of the two components is correctly working at time t; if we denote by A the initially working component, and by B the initially inactive component, one of the following mutually exclusive events must occur:

Event	*Probability*
a) Component A functions until t	$R_1\ (t)$
b) Component A functions until $\tau < t$,	$R_1\ (\tau)$
fails in $(\tau, \tau + d\tau)$,	$dQ_1(\tau)$
Component B has not failed until τ	$R_2(\tau)$
powered, it continuously functions until t	$R_3(t-\tau)$.

The following probabilities are related to the above events:

$$Pr(a) = R_1(t) = exp\,[-\int_0^t(\lambda_o + k_1 t)dt] = exp\,[-\lambda_o t - \frac{k_1}{2}t^2]$$

$$Pr(b) = R_1(\tau)dQ_1(\tau)R_2(\tau)R_3(t-\tau)$$

since the events of b) are statistically independent of each other.

The single factors of the above formula have the following expressions:

$$R_1(\tau)dQ_1(\tau) = f_1(\tau)d\tau = -\frac{dR_1(\tau)}{d\tau}d\tau = (\lambda_o + k_1\tau)\,exp\,[-\lambda_o\tau - \frac{k_1}{2}\tau^2]\,d\tau$$

$$R_2(\tau) = exp\,[-\int_0^\tau(\lambda_1 + k_2 t)\,dt] = exp\,[-\lambda_1\tau - \frac{k_2}{2}\tau^2]$$

$$R_3(t-\tau) = exp\,(-\int_0^{t-\tau}[(\lambda_o + k_2\tau) + k_1(t-\tau)]\,d(t-\tau)) = exp\,[-\lambda_o(t-\tau) - k_2\tau(t-\tau) - \frac{k_1}{2}(t-\tau)^2]$$

Since

$$R_{Seq}(t) = Pr(a) + Pr(b)$$

it follows that

$$R_{Seq}(t) = R_1(t) + \int_0^t f_1(\tau)R_2(\tau)R_3(t-\tau)\,d\tau$$

The development of this fomula is shown in Appendix D; the final result is:

$$(12)\quad R_{Seq}(t) = \frac{3k_1 - k_2}{2k_1 - k_2}\,exp\,(-\lambda_o t - \frac{k_1}{2}t^2) - \frac{k_1}{2k_1 - k_2}\,exp\,[-(\lambda_o + \lambda_1)t - \frac{k_1 + k_2}{2}t^2] + [\lambda_o(2k_1 - k_2) - \lambda_1 k_1 - (k_2 - k_1)k_1 t] \cdot \sqrt{\frac{\pi}{2(2k_1 - k_2)^2}} \cdot exp\,[-\lambda_o t - \frac{(k_1^2 - k_2^2 + k_1 k_2)t^2}{2(2k_1 - k_2)} + \frac{\lambda_1[\lambda_1 + 2(k_2 - k_1)t]}{2(2k_1 - k_2)}] \cdot [\Phi(\frac{\lambda_1 - k_1 t}{\sqrt{2(2k_1 - k_2)}}) - \Phi(\frac{\lambda_1 + (k_2 - k_1)t}{\sqrt{2(2k_1 - k_2)}})]$$

Fig. 13 shows the behavior of $R_{seq}(t)$ for values of parameters chosen on the basis of the following considerations:

– the active component has the same values used in the operative redundancy example for the fully stressed component;

– for the initially inactive component it is assumed that both λ and k have a value a hundred times smaller than that of the active component. This assumption is analogous to that of λ for commonly used electronic components.

It follows that:

$$\lambda_o = 5 \cdot 10^{-7};\ \lambda_1 = 5 \cdot 10^{-9};$$
$$k_1 = 10^{-8},\ 4 \cdot 10^{-8},\ 8 \cdot 10^{-8}$$
$$k_2 = 10^{-2}\ k_1 = 10^{-10},\ 4 \cdot 10^{-10},\ 8 \cdot 10^{-10}.$$

As a reference, in the same Figure the behavior of a non-redundant single block reliability is shown. The values of the parameter for this reference are obviously identical to those relating to the example of Fig. 10.

3.5. COMPARISON BETWEEN THE TWO REDUNDANT CONFIGURATIONS AND THE NON-REDUNDANT BLOCK

In order to make a comparison between the redundant configurations examined in the previous paragraphs, Fig. 14 shows the reliability behavior of these configurations and of the single non-redundant block. The values of the parameters are chosen in order to permit a valid comparison, i.e. the operative redundant configuration must be compared with that standby redundant configuration which has

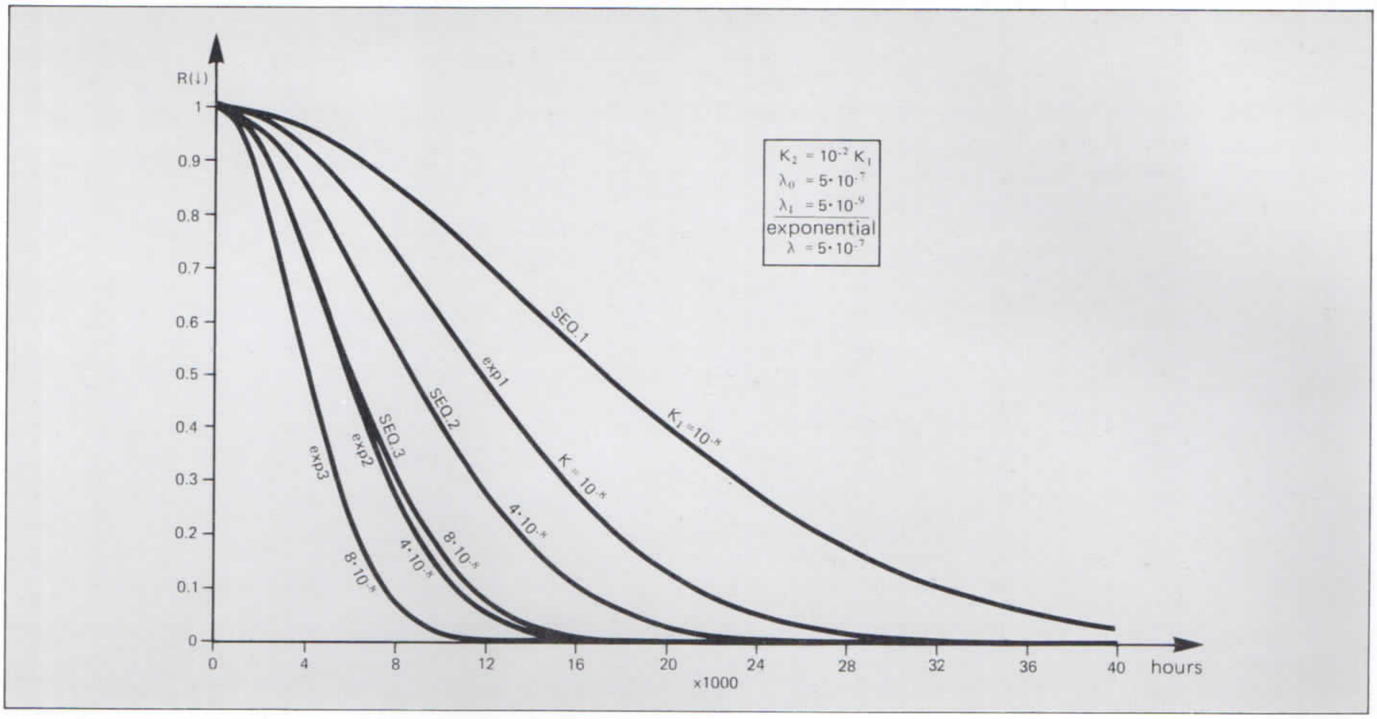


Fig. 13 Standby Redundancy Configuration Reliability Behavior

a k_1 value equal to its own k_2 value. This is done in order to take into account the different stress conditions to which the blocks in the different system configurations are subjected.

The behavior of the single non-redundant block is used as reference for the redundant configurations having the same values of parameters for the fully stressed block. From this figure it can be seen that no definite conclusion can be reached as to which of the two redundant configurations is better from the reliability point of view, since the behavior of the curves is strongly influenced by the effects of the stressed variations on the parameters.

Therefore, the comparison between the configurations which may be represented by the model examined here must be made following a careful evaluation of the parameters.

APPENDIX A
Justification of formula (6)

For the calculation of

$$R(t) = exp\left\{-\int_{t_i}^{t_f} [\lambda_o + k(t-t_1)u(t-t_1)]\, dt\right\}$$

the relevant positions of instants t_i, t_f, t_1, must be considered. There are 3 cases:

a) $t_1 < t_i < t_f$

In this case

$$R(t) = exp\left\{-\int_{t_i}^{t_f} \lambda_o dt - \int_{t_i}^{t_f} k(t-t_1)\, d(t-t_1)\right\} =$$

$$= exp\left\{-\lambda_o(t_f - t_i) - \frac{1}{2}k(t-t_1)^2 \Big|_{t_i}^{t_f}\right\} =$$

$$= exp\left\{-[\lambda_o + k(t_i - t_1) + \frac{1}{2}k t_m]\, t_m\right\}$$

where $t_m = t_f - t_i$ = mission duration

b) $t_i < t_f < t_1$

This is an obvious case, since the increase has not yet begun. Therefore:

$$R(t) = exp\left\{-\int_{t_i}^{t_f} \lambda_o dt\right\} = exp(-\lambda_o t_m)$$

c) $t_i < t_1 < t_f$

In this case:

$$R(t) = exp\left\{-\int_{t_i}^{t_f} \lambda_o dt - \int_{t_1}^{t_f} k(t-t_1)\, d(t-t_1)\right\} =$$

$$= exp\left\{-\lambda_o(t_f - t_i) - \frac{1}{2}k(t-t_1)^2 \Big|_{t_1}^{t_f}\right\} =$$

$$= exp\left\{-[\lambda_o t_m + \frac{1}{2}k(t_f - t_1)^2\right\}$$

The results of cases a), b) and c) can be presented in the single formula:

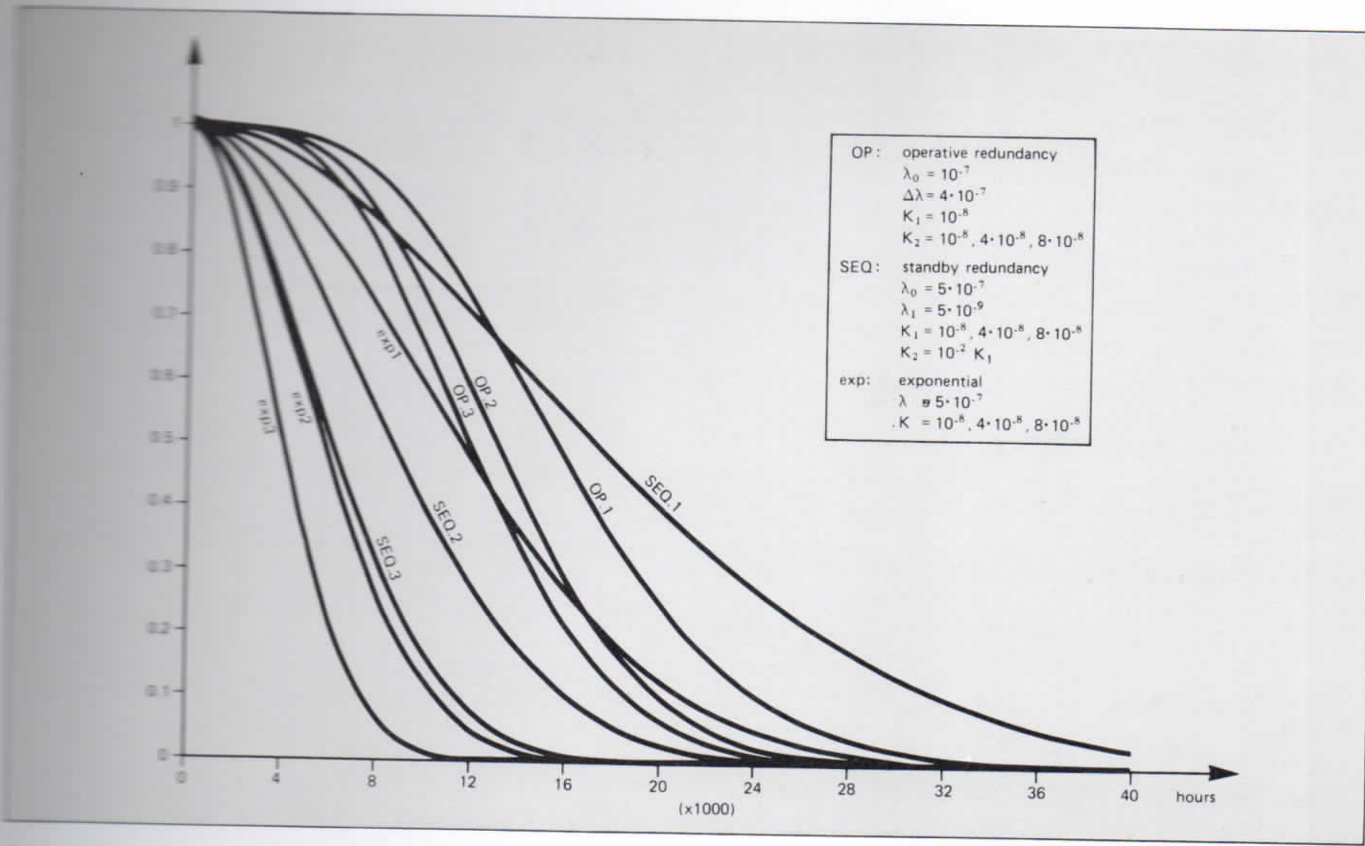


Fig. 14 Comparison Between Redundant Configurations

(A-1) $$R(t) = exp \left\{ - \left\{ \lambda_o + [k(t_i - t_1) + \frac{1}{2} k t_m] \, u(t_i - t_1) \right\} t_m - \frac{1}{2} k (t_f - t_1)^2 \, u(t_f - t_1) \, u(t_1 - t_i) \right\}$$

where

$$u(t_i - t_1) = \begin{cases} 0 & \text{for } t_i \leqslant t_1 \\ 1 & \text{for } t_i > t_1 \end{cases}$$

$$u(t_1 - t_i) = \begin{cases} 0 & \text{for } t_i > t_1 \\ 1 & \text{for } t_i \leqslant t_1 \end{cases}$$

APPENDIX B
Justification of formula (9)

For the calculation of

$$R(t_m = t_f - t_i) = exp \left\{ - \int_{t_i}^{t_f} z(t) \, dt \right\}$$

the relevant positions of the instants t_s ($s = 1, 2, \ldots, n$) and t_i must be considered.

In the case of 3 points of slope variation, one has:

$$R(t_m = t_f - t_i) =$$

$$= exp \left\{ - [\int_{t_i}^{t_f} \lambda_o \, dt + \int_{Max(t_i, t_1)}^{t_f} k_1 (t - t_1) \, u(t - t_1) \, dt + \int_{Max(t_i, t_2)}^{t_f} (k_2 - k_1)(t - t_2) \, u(t - t_2) \, dt + \int_{Max(t_i, t_3)}^{t_f} (k_3 - k_2)(t - t_3) \, u(t - t_3) \, dt] \right\}$$

There are no problems for the first integral. For the calculation of the successive integrals, the first of them can be taken as an example. The following two cases must be considered, depending on $t_i \lesseqgtr t_1$:

a) $t_i < t_1$

Since for $t_i \leqslant t < t_1$, the function $u(t - t_1) = 0$, the integral gives:

$$\int_{t_1}^{t_f} k_1 (t - t_1) \, d(t - t_1) = \frac{1}{2} k_1 (t_f - t_1)^2$$

b) $t_i > t_1$

In this case the integral to be calculated is

$$\int_{t_i}^{t_f} k_1 (t - t_1) \, d(t - t_1) = \frac{1}{2} k_1 (t - t_1)^2 \Big|_{t_i}^{t_f} =$$

$$= [k_1 (t_i - t_1) + \frac{1}{2} k_1 t_m] \, t_m$$

where $t_m = t_f - t_i$.

The two formulae relevant to the cases a) and b) can be grouped together in

$$\int_{Max(t_i,t_1)}^{t_f} k_1(t-t_1)\,u(t-t_1)\,dt =$$
$$= \frac{1}{2} k_1(t_f-t_1)^2 u(t_1-t_i) +$$
$$+ [k_1(t_i-t_1) + \frac{1}{2} k_1 t_m]\, t_m u(t_i-t_1)$$

In the considered case of 3 points of slope variation, therefore:

$$R(t_m) = exp\left\{-\left\{\lambda_o t_m + \frac{1}{2} k_1(t_f-t_1)^2 u(t_1-t_i) + \right.\right.$$
$$+ \frac{1}{2}(k_2-k_1)(t_f-t_2)^2 u(t_2-t_i) +$$
$$+ \frac{1}{2}(k_3-k_2)(t_f-t_3)^2 u(t_3-t_i) +$$
$$+ [k_1(t_i-t_1) + \frac{1}{2} k_1 t_m]\, t_m\, u(t_i-t_1) +$$
$$+ [(k_2-k_1)(t_i-t_2) + \frac{1}{2}(k_2-k_1) t_m] t_m u(t_i-t_2) +$$
$$\left.\left. + [(k_3-k_2)(t_i-t_3) + \frac{1}{2}(k_3-k_2) t_m] t_m u(t_i-t_3)\right\}\right\}$$

The generalization of this formula to the case of n points of slope variation gives formula (9); the conditions on the step functions are introduced to eliminate the discontinuity points in $t_i = t_s$.

APPENDIX C
Justification of formula (11)

The substitution in the expression $R_{op}(t)$ of expressions $R_1(t)$, $f_1(\tau)$, $R_2(t-\tau)$ given in the text gives:

$$R_{op}(t) = exp(-2\lambda_o t - k_1 t^2) +$$
$$+ 2\int_0^t (\lambda_o + k_1\tau)\, exp(-\lambda_o\tau - \frac{k_1}{2}\tau^2)\cdot$$
$$\cdot\, exp(-\lambda_o\tau - \frac{k_1}{2}\tau^2)\cdot$$
$$\cdot\, exp[-(\lambda_o+\Delta\lambda)(t-\tau) - k_1\tau(t-\tau) - \frac{k_2}{2}(t-\tau)^2]\,d\tau$$

By performing the calculations on the exponents, one obtains

$$R_{op}(t) = exp(-2\lambda_o t - k_1 t^2) + 2exp\,[-(\lambda_o+\Delta\lambda)t - \frac{k_2}{2} t^2]\cdot$$
$$\cdot\left\{\lambda_o \int_0^t exp\left\{-\frac{k_2}{2}\tau^2 - [\lambda_o - \Delta\lambda + (k_1-k_2)t]\,\tau\right\}d\tau + \right.$$
$$\left. + k_1 \int_0^t \tau\, exp\left\{-\frac{k_2}{2}\tau^2 - [\lambda_o - \Delta\lambda + (k_1-k_2)t]\,\tau\right\}d\tau\right\}$$

For the calculation of the integrals contained in this expression, one proceeds as follows:

Page 307 of "I.S. Gradshteyn & I.M. Ryzhik – Table of Integral Series and Products – Academic Press, 1965" contains:

$$\int_u^\infty exp(-\frac{x^2}{4\beta} - \gamma x)dx =$$
$$= \sqrt{\pi\beta}\, exp(\beta\gamma^2)\,[1 - \Phi(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}})], \quad [Re\beta > 0,\ u > 0]$$

$$\int_0^\infty exp(-\frac{x^2}{4\beta} - \gamma x)dx =$$
$$= \sqrt{\pi\beta}\, exp(\beta\gamma^2)\,[1 - \Phi(\gamma\sqrt{\beta})], \quad [Re\beta > 0]$$

The difference between the above two expressions gives

(C-1)
$$\int_0^u exp(-\frac{x^2}{4\beta} - \gamma x)\,dx =$$
$$= \sqrt{\pi\beta}\, exp(\beta\gamma^2)\,[\Phi(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}) - \Phi(\gamma\sqrt{\beta})],$$
$$[Re\beta > 0,\ u > 0]$$

which is of the same type as the first integral of the formula to be developed.

Consider now

$$\int_0^u x\, exp(-\frac{x^2}{4\beta} - \gamma x)\,dx =$$
$$= \int_0^u x\, exp(-\frac{x^2}{4\beta})\, exp(-\gamma x)\,dx, \quad [Re\beta > 0,\ u > 0]$$

Since

$$D[exp(-\frac{x^2}{4\beta})] = -\frac{x}{2\beta}\, exp(-\frac{x^2}{4\beta})$$

the integral under examination can be written as:

$$-2\beta\int_0^u D\,[exp\,(-\frac{x^2}{4\beta})]\; exp(-\gamma x)\,dx$$

and, integrating by parts:

$$-2\beta\left\{[exp(-\frac{x^2}{4\beta} - \gamma x)]\,\Big|_0^u + \gamma\int_0^u exp(-\frac{x^2}{4\beta} - \gamma x)\,dx\right\}$$

Taking into account the integral C1, the following formula is obtained

(C-2)
$$\int_0^u x\, exp(-\frac{x^2}{4\beta} - \gamma x)\,dx =$$
$$= 2\beta - 2\beta\, exp(-\frac{u^2}{4\beta} - \gamma u) - 2\beta\gamma\sqrt{\pi\beta}\, exp(\beta\gamma^2)\cdot$$
$$\cdot\,[\Phi(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}) - \Phi(\gamma\sqrt{\beta})]$$
$$[Re\beta > 0,\ u > 0]$$

The linear combination of the two integrals (C1) and (C2) with coefficients λ_o and k_1 gives:

$$(C\text{-}3)\quad \lambda_o \int_0^u exp(-\frac{x^2}{4\beta} - \gamma x)\,dx + k_1 \int_0^u x\, exp(-\frac{x^2}{4\beta} - \gamma x)\,dx =$$
$$= 2\beta k_1 - 2\beta k_1 exp(-\frac{u^2}{4\beta} - \gamma u) +$$
$$+ (\lambda_o - 2\beta\gamma k_1)\sqrt{\pi\beta}\, exp(\beta\gamma^2) \cdot$$
$$\cdot [\Phi(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}) - \Phi(\gamma\sqrt{\beta})],$$
$$[Re\,\beta > 0,\ u > 0]$$

In the case under consideration

$$-\frac{1}{4\beta} = -\frac{k_2}{2};$$
$$-\gamma = -[\lambda_o - \Delta\lambda + (k_1 - k_2)t];$$

there are $\beta = \frac{1}{2k_2}$, which satisfies the condition $Re\,\beta > 0$;

$$\gamma = \lambda_o - \Delta\lambda + (k_1 - k_2)\,t;$$

$u = t$, which satisfies the condition $u > 0$. Substitution of the values of β, γ and u in (C3) gives

$$\int_0^t (\lambda_o + k_1\tau)\, exp\{-\frac{k_2}{2}\tau^2 - [\lambda_o - \Delta\lambda + (k_1 - k_2)t]\tau\}\,d\tau =$$
$$= \frac{k_1}{k_2} - \frac{k_1}{k_2} exp[-(k_1 - \frac{k_2}{2})t^2 - (\lambda_o - \Delta\lambda)t] +$$
$$+ \frac{k_1\Delta\lambda + (k_2 - k_1)(\lambda_o + k_1 t)}{k_2}.$$
$$\cdot \sqrt{\frac{\pi}{2k_2}}\, exp\{\frac{[\lambda_o - \Delta\lambda + (k_1 - k_2)t]^2}{2k_2}\}\,[\Phi(\frac{\lambda_o - \Delta\lambda + k_1 t}{\sqrt{2k_2}}) -$$
$$- \Phi(\frac{\lambda_o - \Delta\lambda + (k_1 - k_2)t}{\sqrt{2k_2}})]$$

Introduction of this integral in the expression $R_{op}(t)$ gives

$$R_{op}(t) = exp(-2\lambda_o t - k_1 t^2) +$$
$$+ \frac{2k_1}{k_2} exp\{-(\lambda_o + \Delta\lambda)t - \frac{k_2}{2}t^2\} -$$
$$- \frac{2k_1}{k_2} exp[-\lambda_o t - \Delta\lambda t - \frac{k_2}{2}t^2 + \frac{k_2}{2}t^2 - k_1 t^2 - \lambda_o t + \Delta\lambda t] +$$
$$+ 2\frac{k_1\Delta\lambda + (k_2 - k_1)(\lambda_o + k_1 t)}{k_2}\sqrt{\frac{\pi}{2k_2}}\, exp[-(\lambda_o + \Delta\lambda)t -$$
$$- \frac{k_2}{2}t^2 + \frac{(\lambda_o - \Delta\lambda + k_1 t - k_2 t)^2}{2k_2}] \cdot [\Phi(\frac{\lambda_o - \Delta\lambda + k_1 t}{\sqrt{2k_2}}) -$$
$$- \Phi(\frac{\lambda_o - \Delta\lambda + (k_1 - k_2)t}{\sqrt{2k_2}})]$$

Development of the calculation therefore gives formula (11).

APPENDIX D
Justification of formula (12)

The substitution in the expression $R_{Seq}(t)$ of expressions $R_1(t)$, $f_1(\tau)$, $R_2(\tau)$, $R_3(t - \tau)$ given in the text gives:

$$R_{Seq}(t) = exp(-\lambda_o t - \frac{k_1}{2}t^2) +$$
$$+ \int_0^t (\lambda_o + k_1\tau)\, exp(-\lambda_o\tau - \frac{k_1}{2}\tau^2) \cdot$$
$$\cdot exp(-\lambda_1\tau - \frac{k_2}{2}\tau^2) \cdot$$
$$\cdot exp[-\lambda_o(t - \tau) - k_2\tau(t - \tau) - \frac{k_1}{2}(t - \tau)^2]\,d\tau$$

By performing the calculations on the exponents, one obtains

$$R_{Seq}(t) = exp(-\lambda_o t - \frac{k_1}{2}t^2) \cdot$$
$$\cdot \{1 + \int_0^t (\lambda_o + k_1\tau)\, exp\{-(k_1 - \frac{k_2}{2})\tau^2 -$$
$$- [\lambda_1 + (k_2 - k_1)t]\,\tau\}\,d\tau\}$$

The integral of this formula is of the same type as (C3), where

$$\beta = \frac{1}{2(2k_1 - k_2)};$$
$$\gamma = \lambda_1 + (k_2 - k_1)t;$$
$$u = t.$$

The condition $u > 0$ is always verified; the condition $Re\beta > 0$ is verified when

$$Re\beta > 0 \quad \text{for } k_1 > \frac{k_2}{2}$$

which is always verified if the significance of k_1 and k_2 is considered.

The substitution of expressions β, γ and u in (C3) gives

$$\int_0^t (\lambda_o + k_1\tau)\, exp\{-(k_1 - \frac{k_2}{2})\tau^2 - [\lambda_1 + (k_2 - k_1)t]\,\tau\}\,d\tau =$$
$$= \frac{k_1}{2k_1 - k_2} - \frac{k_1}{2k_1 - k_2} exp[-\frac{(2k_1 - k_2)t^2}{2} -$$
$$- \lambda_1 t - (k_2 - k_1)t^2] +$$
$$+ \frac{\lambda_o(2k_1 - k_2) - \lambda_1 k_1 - (k_2 - k_1)k_1 t}{(2k_1 - k_2)}.$$
$$\cdot \sqrt{\frac{\pi}{2(2k_1 - k_2)}}\, exp[\frac{[\lambda_1 + (k_2 - k_1)t]^2}{2(2k_1 - k_2)}] \cdot$$
$$\cdot [\Phi(\frac{\lambda_1 + k_1 t}{\sqrt{2(2k_1 - k_2)}}) - \Phi(\frac{\lambda_1 + (k_2 - k_1)t}{\sqrt{2(2k_1 - k_2)}})]$$

Introduction of this integral in the expression of $R_{seq}(t)$ gives:

$$R_{Seq}(t) = (1 + \frac{k_1}{2k_1 - k_2}) \, exp\,(-\lambda_o t - \frac{k_1}{2} t^2) -$$

$$- \frac{k_1}{2k_1 - k_2} \, exp\,[-\lambda_o t - \frac{k_1}{2} t^2 - \frac{(2k_1 - k_2)t^2}{2} - \lambda_1 t -$$

$$- (k_2 - k_1) t^2] + [\lambda_o (2k_1 - k_2) - \lambda_1 k_1 - (k_2 - k_1) k_1 t] \cdot$$

$$\cdot \sqrt{\frac{\pi}{2(2k_1 - k_2)^3}} \, exp\,[-\lambda_o t - \frac{k_1}{2} t^2 + \frac{(\lambda_1 + k_2 t - k_1 t)^2}{2(2k_1 - k_2)}] \cdot$$

$$\cdot [\Phi(\frac{\lambda_1 + k_1 t}{\sqrt{2(2k_1 - k_2)}}) - \Phi(\frac{\lambda_1 + (k_2 - k_1) t}{\sqrt{2(2k_1 - k_2)}})]$$

Development of the calculation therefore gives formula (12).

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